

NON-EXISTENCE FOR FRACTIONALLY DAMPED FRACTIONAL DIFFERENTIAL PROBLEMS *

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Abstract In this paper, we are concerned with a fractional differential inequality containing a lower order fractional derivative and a polynomial source term in the right hand side. A non-existence of non-trivial global solutions result is proved in an appropriate space by means of the test-function method. The range of blow up is found to depend only on the lower order derivative. This is in line with the well-known fact for an internally weakly damped wave equation that solutions will converge to solutions of the parabolic part.

Key words Nonexistence, global solution, fractional differential equation, Riemann-Liouville fractional integral and fractional derivative

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1 Introduction

In this paper we consider the problem

$$\begin{cases} D_0^\alpha y(t) + D_0^\beta y(t) = f(t, y(t)), & t > 0, \\ I_0^{1-\alpha} y(t)|_{t=0} = b, \end{cases} \quad (1.1)$$

where D_0^σ is the Riemann-Liouville fractional derivative of order $\sigma > 0$, $0 < \beta \leq \alpha \leq 1$.

A nonexistence result of non-trivial global solutions for the problem (1.1) will be proved when

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$f(t, y(t)) \geq t^\gamma |y(t)|^m$ for some $m > 1$ and $\gamma \in \mathbb{R}$. That is we consider the problem:

$$\begin{cases} D_0^\alpha y(t) + D_0^\beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \\ I_0^{1-\alpha} y(t)|_{t=0} = b, \end{cases} \quad (1.2)$$

where $0 < \beta \leq \alpha \leq 1$ and show that no solutions can exist for all time for certain values of γ and m . In particular, we find the range of values of m for which solutions do not exist globally. Clearly, sufficient conditions for nonexistence provide necessary conditions for existence of solutions.

The interest to fractional calculus has been accelerated the past three decades after the publication of the three papers of Bagley and Torvik [3–5] and the paper by Podlubny [28]. Many phenomena in diverse fields of science and engineering can be described by differential equations of non-integer order. Namely, they arise naturally in viscoelasticity, porous media, electrochemistry, control and electromagnetic, etc [25–27].

In fact it has been shown by experiments that derivatives of non-integer order can describe many phenomena better than derivatives of integer order specially hereditary phenomena and processes.

Some recent applications arose in viscoelasticity, rheology, control systems, synthesis, robots and nanotechnology, etc (see [11, 14, 19, 20, 22, 23, 29]).

Regarding the existence of solutions for various classes of fractional differential equations, there are many results (e.g. see [1, 2, 7–9, 13, 24, 31]). For the issue of nonexistence of solutions for fractional differential equations, we refer to [10, 12, 21, 30] and to [15–18] for partial differential equations involving fractional derivatives (see also references therein).

The existence and uniqueness of solutions for problem (1.1) has been discussed in [14].

In case $\alpha = \beta = 1$ and $f(t, y(t)) = 2y^m(t)$ in (1.1) we obtain

$$\begin{cases} y'(t) = y^m(t), \\ y(t)|_{t=0} = b. \end{cases}$$

This problem has, for $m > 1$, the solution

$$y(t) = [(1-m)(t+c)]^{1/(1-m)},$$

where

$$c = \frac{b^{1-m}}{1-m}.$$

Observe that, for $m > 1$, the solution blows-up in finite time.

When $\alpha = 1$, $\beta = 0$ and $\gamma = 0$, the problem (1.2) with an equality instead of inequality is equivalent to the Bernoulli differential problem

$$\begin{cases} y'(t) + y(t) = y^m(t), & t > 0, \\ y(t)|_{t=0} = b. \end{cases} \quad (1.3)$$

The solution of (1.3) is given by

$$y(t) = [1 + (b^{1-m} - 1) \exp(m-1)t]^{1/(1-m)}.$$

Clearly $y(t)$ blows up in the finite time

$$c = \frac{1}{1-m} \ln(1 - b^{1-m}), \quad m, b > 1.$$

In case $\alpha = \beta$ in (1.2) we obtain the problem with only one fractional derivative

$$\begin{cases} 2D_0^\alpha y(t) \geq t^\gamma |y(t)|^m, & t > 0, \\ I_0^{1-\alpha} y(t)|_{t=0} = b. \end{cases} \quad (1.4)$$

Problem (1.4) has been considered by Laskri and Tatar [21]. It was shown that if $\gamma > -\alpha$ and $1 < m \leq \frac{\gamma+1}{1-\alpha}$, then, Problem (1.4) does not admit global nontrivial solutions when $b \geq 0$.

Here, we would like to investigate the case where a lower order fractional derivative is present in the equation (or inequality). It is known that for hyperbolic equations, say the wave equation with an internal fractional damping represented by the first derivative (i.e. $\alpha = 2, \beta = 1$ also known as the Telegraph equation), this damping has a dissipation effect. It will compete with the polynomial source and may take it over this blowing-up term under certain circumstances. Moreover, it has been shown for the telegraph problem that solutions approach the solution of the same problem without the highest derivative when t goes to infinity (that is the parabolic equation). This result has been generalized to the fractional derivative case in [6] and in [30]. For our problem here (1.2), we would like to see how much influential $D_0^\beta y$ will be on the blow-up phenomenon. In particular, how the range of values m ensuring blow-up in finite time would be affected. We reached the conclusion that here also it is the lower order derivative (i.e. β) which determines the range of blow-up just like the parabolic part in the hyperbolic problem. The rest of the paper is divided into two sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. Section 3 is devoted to the nonexistence result.

2 Preliminaries

In this section we present some definitions, lemmas, properties and notation which will be used in our result later.

Definition 2.1 *The Riemann-Liouville left-sided fractional integral $I_a^\alpha f$ of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a, \quad \alpha > 0, \quad (2.1)$$

provided that the integral exists. Here $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I_a^0 f = f$.

Definition 2.2 *The Riemann-Liouville right-sided fractional integral $I_{b-}^\alpha f$ of order $\alpha > 0$ is defined by*

$$I_{b-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t < b, \quad \alpha > 0, \quad (2.2)$$

provided that the integral exists. When $\alpha = 0$, we define $I_{b-}^0 f = f$.

Definition 2.3 *The Riemann-Liouville left-sided fractional derivative $D_a^\alpha f$ of order α , $0 < \alpha < 1$, is defined by*

$$D_a^\alpha f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t),$$

that is,

$$D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds, \quad t > a, \quad 0 < \alpha < 1, \quad (2.3)$$

when $\alpha = 1$ we have $D_a^\alpha f = Df$. In particular, when $\alpha = 0$, $D_a^0 f = f$.

Definition 2.4 The Riemann-Liouville right-sided fractional derivative $D_{b-}^\alpha f$ of order α , $0 < \alpha < 1$, is defined by

$$D_{b-}^\alpha f(t) = -\frac{d}{dt} I_{b-}^{1-\alpha} f(t),$$

that is,

$$D_{b-}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^\alpha} ds, \quad t < b, \quad 0 < \alpha < 1. \quad (2.4)$$

In particular, when $\alpha = 0$, $D_{b-}^\alpha f = f$.

Lemma 2.5 (Fractional Integration by Parts) Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\int_a^b \varphi(t) (I_a^\alpha \psi)(t) dt = \int_a^b \psi(t) (I_{b-}^\alpha \varphi)(t) dt. \quad (2.5)$$

Definition 2.6 We consider the weighted spaces of continuous functions

$$C_\gamma[a, b] = \{f : (a, b] \rightarrow \mathbb{R} : (t-a)^\gamma f(t) \in C[a, b]\}, \quad 0 < \gamma < 1,$$

$$C_0[a, b] = C[a, b],$$

and

$$C_{1-\alpha}^\alpha[a, b] = \{f \in C_{1-\alpha}[a, b] : D_a^\alpha f \in C_{1-\alpha}[a, b]\}, \quad 0 < \alpha < 1. \quad (2.6)$$

Lemma 2.7 Let $0 \leq \gamma < 1$ and $f \in C_\gamma[a, b]$. Then

$$I_a^\alpha f(a) = \lim_{t \rightarrow a} I_a^\alpha f(t) = 0, \quad 0 \leq \gamma < \alpha.$$

Proof Since $f \in C_\gamma[a, b]$ then $(t-a)^\gamma f(t)$ is continuous on $[a, b]$ and on $[a, b]$ we have

$$|(t-a)^\gamma f(t)| < M,$$

for some positive constant M . Therefore

$$|I_a^\alpha f(t)| < M \left[I_a^\alpha (s-a)^{-\gamma} \right](t) = M \frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)} (t-a)^{\alpha-\gamma}.$$

As $\alpha > \gamma$ we see that

$$I_a^\alpha f(a) = \lim_{t \rightarrow a} I_a^\alpha f(t) = 0, \quad 0 \leq \gamma < \alpha$$

which completes the proof of Lemma 2.7.

Lemma 2.8 Let $\varphi \in C^1[0, \infty)$ be a test function, that is: $\varphi(t) \geq 0$, $\varphi(t)$ is non-increasing and such that

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2] \\ 0, & t \in [T, \infty), \end{cases}$$

for $T > 0$. Then

$$I(T) = \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^p} \right)^m (t) dt \leq K_{\alpha,m} T^{1-\alpha m}, \quad 0 < \alpha < 1, T, p, m > 0 \quad (2.7)$$

where

$$K_{\alpha,m} = \frac{K_1^m}{2^{m(1-\alpha)+1} \Gamma^m(2-\alpha) [m(1-\alpha) + 1]}, \quad (2.8)$$

and K_1 is a bound for $\frac{|\varphi'(r)|}{\varphi(r)^p}$.

Proof Using (2.2), we see that

$$I(T) = \int_{T/2}^T \left(\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^p} ds \right)^m dt. \quad (2.9)$$

The change of variable $\sigma T = t$ in (2.9) yields

$$I(T) = \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)} \int_{\sigma T}^T (s-\sigma T)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^p} ds \right)^m T d\sigma. \quad (2.10)$$

Another change of variable $s = rT$ in (2.10) gives

$$\begin{aligned} I(T) &= \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)} \int_{\sigma}^1 (rT - \sigma T)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^p} dr \right)^m T d\sigma \\ &= \frac{T^{1-\alpha m}}{\Gamma^m(1-\alpha)} \int_{1/2}^1 \left(\int_{\sigma}^1 (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^p} dr \right)^m d\sigma. \end{aligned} \quad (2.11)$$

Since $\varphi \in C^1([0, \infty))$, we may assume without loss of generality that

$$\frac{|\varphi'(r)|}{\varphi(r)^p} \leq K_1,$$

for some positive constant K_1 , for otherwise we consider $\varphi^\lambda(r)$ with some sufficiently large λ .

Therefore from (2.11) we get

$$\begin{aligned} I(T) &\leq \frac{K_1^m T^{1-\alpha m}}{\Gamma^m(1-\alpha)} \int_{1/2}^1 \left(\int_{\sigma}^1 (r-\sigma)^{-\alpha} dr \right)^m d\sigma = \frac{K_1^m T^{1-\alpha m}}{\Gamma^m(2-\alpha)} \int_{1/2}^1 (1-\sigma)^{m(1-\alpha)} d\sigma \\ &= \frac{K_1^m}{2^{m(1-\alpha)+1} \Gamma^m(2-\alpha) [m(1-\alpha) + 1]} T^{1-\alpha m}. \end{aligned}$$

Therefore

$$I(T) \leq K_{\alpha,m} T^{1-\alpha m}.$$

Remark 2.9 Lemma 2.8 is true also for the case $\alpha = 1$. We prove this fact in the following lemma.

Lemma 2.10 Let φ be as in Lemma 2.8. Then

$$I(T) = \int_{T/2}^T \left(\frac{|\varphi'(t)|}{\varphi^p(t)} \right)^m dt \leq \frac{1}{2} K_1^m T^{1-m}, \quad T, p, m > 0, \quad (2.12)$$

with

$$\frac{|\varphi'(r)|}{\varphi(r)^p} \leq K_1.$$

Proof The change of variable $sT = t$ in the expression of $I(T)$ leads to

$$I(T) = \int_{1/2}^1 \left(\frac{|\varphi'(s)|}{T\varphi^p(s)} \right)^m T ds = T^{1-m} \int_{1/2}^1 \left(\frac{|\varphi'(s)|}{\varphi^p(s)} \right)^m ds \leq \frac{1}{2} K_1^m T^{1-m}.$$

3 Nonexistence result

In this section, we consider the problem

$$\begin{cases} D_0^\alpha y(t) + D_0^\beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \quad m > 1, \quad 0 < \beta < \alpha < 1, \\ I_0^{1-\alpha} y(t)|_{t=0} = b, \end{cases} \quad (3.1)$$

where D_0^σ is defined in (2.3). Nonexistence of non-trivial solutions is investigated in the space $C_{1-\alpha}^\alpha$ defined in (2.6).

Theorem 3.1 Assume that $\gamma > -\beta$ and $1 < m \leq \frac{\gamma+1}{1-\beta}$. Then, Problem (3.1) does not admit global nontrivial solutions in $C_{1-\alpha}^\alpha$, when $b \geq 0$.

Proof Assume, on the contrary, that a nontrivial solution y exists for all time $t > 0$. Let φ be as in Lemma 2.8. Multiplying the inequality in (3.1) by $\varphi(t)$ and integrating over $(0, T)$ we get

$$I_1 := \int_0^T t^\gamma |y(t)|^m \varphi(t) dt \leq \int_0^T D_0^\alpha y(t) \varphi(t) dt + \int_0^T D_0^\beta y(t) \varphi(t) dt. \quad (3.2)$$

Let

$$I_2 := \int_0^T \varphi(t) D_0^\alpha y(t) dt,$$

and

$$I_3 := \int_0^T \varphi(t) D_0^\beta y(t) dt.$$

From the definition of $D_0^\alpha y$ in (2.3) we can write

$$I_2 = \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\alpha} y(t) dt.$$

An integration by parts yields

$$I_2 = [\varphi(t) I_0^{1-\alpha} y(t)]_{t=0}^T - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) dt.$$

Since $\varphi(T) = 0$, $\varphi(0) = 1$ and $I_0^{1-\alpha} y(0) = b$, then

$$I_2 = -b - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) dt.$$

As $b \geq 0$, we have

$$\begin{aligned} I_2 &\leq - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) dt \leq \int_0^T |\varphi'(t)| (I_0^{1-\alpha} |y|)(t) dt \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\alpha} ds dt. \end{aligned} \quad (3.3)$$

Because $\varphi(t)$ is nonincreasing $\varphi(s) \geq \varphi(t)$ for all $t \geq s$, and therefore

$$\frac{1}{\varphi(s)^{1/m}} \leq \frac{1}{\varphi(t)^{1/m}}, \quad m > 1.$$

Also we have

$$\varphi'(t) = 0, \quad t \in [0, T/2].$$

Thus

$$\begin{aligned} I_2 &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \frac{\varphi(s)^{1/m}}{\varphi(s)^{1/m}} ds dt \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} ds dt \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} ds dt \\ &\leq \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \left(I_0^{1-\alpha} \varphi^{1/m} |y| \right) (t) dt. \end{aligned}$$

A fractional integration by parts (2.5), in the last expression yields

$$I_2 \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \varphi(t)^{1/m} |y(t)| dt.$$

Next, we multiply by $t^{\gamma/m} t^{-\gamma/m}$ inside the integral in the right hand side

$$I_2 \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \varphi(t)^{1/m} \frac{t^{\gamma/m}}{t^{\gamma/m}} |y(t)| dt.$$

For $\gamma < 0$ we have $t^{-\gamma/m} < T^{-\gamma/m}$ (because $t < T$) and for $\gamma > 0$ we get $t^{-\gamma/m} < 2^{\gamma/m} T^{-\gamma/m}$ (because $T/2 < t$): that is

$$t^{-\gamma/m} < \max \{1, 2^{\gamma/m}\} T^{-\gamma/m}.$$

Then

$$I_2 \leq \max \{1, 2^{\gamma/m}\} T^{-\gamma/m} \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) t^{\gamma/m} \varphi(t)^{1/m} |y(t)| dt. \quad (3.4)$$

By Hölder's inequality, it is clear that

$$I_2 \leq \max \{1, 2^{\gamma/m}\} T^{-\gamma/m} \left(\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right)^{\frac{1}{m}} \left(\int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)^{m'} (t) dt \right)^{\frac{1}{m'}}.$$

Lemma 2.8 implies that

$$I_2 \leq \max \{1, 2^{\gamma/m}\} T^{-\gamma/m} \left(\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right)^{\frac{1}{m}} \left(K_{\alpha, m'} T^{1-\alpha m'} \right)^{\frac{1}{m'}}, \quad (3.5)$$

where $K_{\alpha, m'}$ is the constant appearing in Lemma 2.8 corresponding to the present exponents. Therefore from (3.5) we have the estimate

$$I_2 \leq \max \{1, 2^{\gamma/m}\} K_{\alpha, m'}^{\frac{1}{m'}} T^{1/m' - \alpha - \gamma/m} I_1^{\frac{1}{m}}. \quad (3.6)$$

Now, we turn to I_3 . First, since $y \in C_{1-\alpha}[0, T]$ and $1-\alpha < 1-\beta$, then by Lemma 2.7 we have

$$I_0^{1-\beta} y(0) = \lim_{t \rightarrow 0} I_0^{1-\beta} y(t) = 0.$$

An integration by parts in

$$I_3 = \int_0^T \varphi(t) D_0^\beta y(t) dt = \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\beta} y(t) dt$$

gives

$$I_3 = \left[\varphi(t) I_0^{1-\beta} y(t) \right]_{t=0}^T - \int_0^T \varphi'(t) I_0^{1-\beta} y(t) dt.$$

Since $\varphi(T) = 0$ and $I_0^{1-\beta} y(0) = 0$, it follows that

$$\begin{aligned} I_3 &= - \int_0^T \varphi'(t) I_0^{1-\beta} y(t) dt \leq \int_0^T |\varphi'(t)| \left(I_0^{1-\beta} |y| \right)(t) dt \\ &\leq \frac{1}{\Gamma(1-\beta)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\beta} ds dt. \end{aligned}$$

Replacing α by β in the argument above allows us to write

$$I_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t) t^{\gamma/m} \varphi(t)^{1/m} |y(t)| dt, \quad (3.7)$$

or simply

$$I_3 \leq K_{\beta, m'}^{\frac{1}{m'}} \max \left\{ 1, 2^{\gamma/m} \right\} T^{1/m' - \beta - \gamma/m} I_1^{\frac{1}{m}}. \quad (3.8)$$

From (3.2), (3.6) and (3.8), we have

$$\begin{aligned} I_1 &\leq \max \left\{ 1, 2^{\gamma/m} \right\} K_{\alpha, m'}^{\frac{1}{m'}} T^{1/m' - \alpha - \gamma/m} I_1^{\frac{1}{m}} + K_{\beta, m'}^{\frac{1}{m'}} \max \left\{ 1, 2^{\gamma/m} \right\} T^{1/m' - \beta - \gamma/m} I_1^{\frac{1}{m}} \\ &\leq \max \left\{ K_{\alpha, m'}^{\frac{1}{m'}}, K_{\beta, m'}^{\frac{1}{m'}} \right\} \max \left\{ 1, 2^{\gamma/m} \right\} \left(T^{1/m' - \alpha - \gamma/m} + T^{1/m' - \beta - \gamma/m} \right) I_1^{\frac{1}{m}}. \end{aligned}$$

Therefore

$$I_1^{\frac{1}{m'}} \leq K_2 \left(T^{1/m' - \alpha - \gamma/m} + T^{1/m' - \beta - \gamma/m} \right), \quad (3.9)$$

with

$$K_2 := \max \left\{ K_{\alpha, m'}^{\frac{1}{m'}}, K_{\beta, m'}^{\frac{1}{m'}} \right\} \max \left\{ 1, 2^{\gamma/m} \right\}.$$

Raising both sides of (3.9) to the power m' we obtain

$$I_1 \leq K_3 \left(T^{1 - \alpha m' - \gamma m'/m} + T^{1 - \beta m' - \gamma m'/m} \right), \quad (3.10)$$

with

$$K_3 = 2^{1-m'} K_2^{m'}.$$

If $m < \frac{\gamma+1}{1-\beta}$ we see that $1 - \beta m' - \gamma m'/m < 0$, $1 - \alpha m' - \gamma m'/m < 0$, and consequently $T^{1-\beta m' - \gamma m'/m} \rightarrow 0$ and $T^{1-\alpha m' - \gamma m'/m} \rightarrow 0$ as $T \rightarrow \infty$. Then, from (3.10), we obtain

$$\lim_{T \rightarrow \infty} I_1 = \lim_{T \rightarrow \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) dt = 0.$$

We reach a contradiction since the solution is not supposed to be trivial.

In the case $m = \frac{\gamma+1}{1-\beta}$ we have $1 - \beta m' - \gamma m'/m = 0$, $1 - \alpha m' - \gamma m'/m \leq 0$, and the relation (3.10) ensures that

$$\lim_{T \rightarrow \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) dt \leq K_4. \quad (3.11)$$

Further, in view of (3.2), (3.4) and (3.7), we see that

$$I_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T t^{\gamma/m} \varphi(t)^{1/m} |y(t)| \left[\left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t) + \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t) \right] dt.$$

Thanks to Hölder's inequality, it is clear that

$$\begin{aligned}
I_1 &\leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \\
&\quad \times \left\{ \int_{T/2}^T \left[\left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) + \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \right]^{m'} dt \right\}^{\frac{1}{m'}} \\
&\leq \max \left\{ 1, 2^{\gamma/m} \right\} 2^{1/m} T^{-\gamma/m} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \\
&\quad \times \left\{ \int_{T/2}^T \left[\left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)^{m'} (t) + \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right)^{m'} (t) \right] dt \right\}^{\frac{1}{m'}}.
\end{aligned}$$

Therefore, by Lemma 2.8, we obtain

$$\begin{aligned}
I_1 &\leq K_5 T^{-\gamma/m} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \left[K_{\alpha, m'} T^{1-\alpha m'} + K_{\beta, m'} T^{1-\beta m'} \right]^{\frac{1}{m'}} \\
&= K_5 \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \left[K_{\alpha, m'} T^{1-\alpha m' - \gamma m'/m} + K_{\beta, m'} T^{1-\beta m' - \gamma m'/m} \right]^{\frac{1}{m'}},
\end{aligned}$$

with

$$K_5 = \max \left\{ 1, 2^{\gamma/m} \right\} 2^{1/m}.$$

Since $m = \frac{\gamma+1}{1-\beta}$, then $1 - \beta m' - \gamma m'/m = 0$ and $1 - \alpha m' - \gamma m'/m \leq 0$. Therefore

$$I_1 \leq K_6 \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}}$$

for some positive constant K_6 , with

$$\lim_{T \rightarrow \infty} \int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt = 0$$

due to the convergence of the integral in (3.11). This is again a contradiction. The proof is complete.

Next, we take $\alpha = 1$ and $0 < \beta < 1$, that is

$$\begin{cases} y'(t) + D_0^\beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \quad m > 1, \quad 0 < \beta < 1, \\ y(t)|_{t=0} = b \in \mathbb{R}. \end{cases} \quad (3.12)$$

Theorem 3.2 Assume that $\gamma > -\beta$ and $1 < m \leq \frac{\gamma+1}{1-\beta}$. Then, Problem (3.12) does not admit global nontrivial solutions when $b \geq 0$.

Proof Assume, on the contrary, that a nontrivial solution y exists for all time $t > 0$. Let φ be as in Lemma 2.8. Multiplying the inequality in (3.12) by $\varphi(t)$ and integrating we get

$$J_1 = \int_0^T t^\gamma |y(t)|^m \varphi(t) dt \leq \int_0^T y'(t) \varphi(t) dt + \int_0^T D_0^\beta y(t) \varphi(t) dt. \quad (3.13)$$

Let

$$J_2 = \int_0^T \varphi(t) y'(t) dt, \quad (3.14)$$

and

$$J_3 = \int_0^T \varphi(t) D_0^\beta y(t) dt. \quad (3.15)$$

Following procedure as in the proof of Theorem **3.1**, we obtain the following estimates for J_2 and J_3

$$J_2 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} |y(t)| \varphi(t)^{1/m} t^{\gamma/m} dt, \quad (3.16)$$

(or By using Hölder's inequality and Lemma **2.10**)

$$J_2 \leq \max \left\{ 1, 2^{\gamma/m} \right\} K_1 T^{1/m' - 1 - \gamma/m} J_1^{\frac{1}{m}}, \quad (3.17)$$

and

$$J_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) t^{\gamma/m} \varphi(t)^{1/m} |y(t)| dt, \quad (3.18)$$

(or By using Hölder's inequality and Lemma **2.8**)

$$J_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} K_{\beta, m'}^{\frac{1}{m'}} T^{1/m' - \beta - \gamma/m} J_1^{\frac{1}{m}}. \quad (3.19)$$

From (3.13), (3.17) and (3.19), we have

$$J_1^{\frac{1}{m'}} \leq K_2 \left(T^{1/m' - 1 - \gamma/m} + T^{1/m' - \beta - \gamma/m} \right), \quad (3.20)$$

with

$$K_2 := \max \left\{ 1, 2^{\gamma/m} \right\} \max \left\{ K_{\beta, m'}^{\frac{1}{m'}}, K_1 \right\}.$$

Raising both sides of (3.20) to the power m' we obtain

$$J_1 \leq K_3 \left(T^{1 - m' - \gamma m' / m} + T^{1 - \beta m' - \gamma m' / m} \right), \quad (3.21)$$

with

$$K_3 = 2^{1 - m'} K_2^{m'}.$$

If $m < \frac{\gamma+1}{1-\beta}$ we see that $1 - m' - \gamma m' / m < 0$, $1 - \beta m' - \gamma m' / m < 0$. Then from (3.21) we obtain

$$\lim_{T \rightarrow \infty} J_1 = \lim_{T \rightarrow \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) dt = 0.$$

We reach a contradiction since the solution is not supposed to be trivial.

In the case $m = \frac{\gamma+1}{1-\beta}$ we have $1 - m' - \gamma m' / m \leq 0$, $1 - \beta m' - \gamma m' / m = 0$, and the relation (3.21) ensures that

$$\lim_{T \rightarrow \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) dt \leq K_4. \quad (3.22)$$

Also from (3.13), (3.16) and (3.18), we have

$$J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T t^{\gamma/m} \varphi(t)^{1/m} |y(t)| \left[\frac{|\varphi'(t)|}{\varphi(t)^{1/m}} + \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \right] dt.$$

By using Hölder's inequality, it is clear that

$$J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \\ \times \left[\int_{T/2}^T \left[\frac{|\varphi'(t)|}{\varphi(t)^{1/m}} + \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t) \right]^{m'} dt \right]^{\frac{1}{m'}}.$$

Therefore, by Lemma 2.8 and Lemma 2.10 and $\varphi \in C^1[0, \infty)$, we have

$$J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \left[K_5 T^{1-m'} + K_6 T^{1-\beta m'} \right]^{1/m'},$$

for some positive constants K_5 and K_6 , and then

$$J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \left[K_5 T^{1-m'-\gamma m'/m} + K_6 T^{1-\beta m'-\gamma m'/m} \right]^{1/m'} \\ \leq \max \left\{ 1, 2^{\gamma/m} \right\} \left[\int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt \right]^{\frac{1}{m}} \left[K_5 T^{1-m'-\gamma m'/m} + K_6 \right]^{1/m'},$$

and

$$\lim_{T \rightarrow \infty} \int_{T/2}^T t^\gamma \varphi(t) |y(t)|^m dt = 0$$

due to the convergence of the integral in (3.22). This is again a contradiction and the proof of Theorem 3.2 is complete.

Finally, we take $\alpha = \beta = 1$, this mean we consider the Cauchy problem

$$\begin{cases} y'(t) \geq t^\gamma |y(t)|^m, & t > 0, \quad m > 1, \\ y(t)|_{t=0} = b \in \mathbb{R}. \end{cases} \quad (3.23)$$

Theorem 3.3 *Assume that $\gamma > -1$ and $m > 1$. Then, Problem (3.23) does not admit global nontrivial solutions when $b \geq 0$.*

Proof Similar to the proof of Theorem 3.1.

Conclusion 3.1 According to Theorems 3.1, 3.2 and having in mind the results in [21] it appears that the addition of the term $D_0^\beta y$, $\beta < \alpha$, does not prevent the nonexistence. However, it does affect the exponent m . The range of m is reduced to $1 < m \leq \frac{\gamma+1}{1-\beta}$ instead of $1 < m \leq \frac{\gamma+1}{1-\alpha}$. This shows that the range does not depend on the highest derivative. It depends on the lowest derivative. This is a well-established result for the Telegraph equation. Indeed, for this problem, it has been proved that solutions approach solutions of the corresponding parabolic part.

In case m is fixed from the beginning then we need $\gamma > m(1-\beta)-1$ instead of $\gamma > m(1-\alpha)-1$.

Therefore, it is the derivative of lower order which determines the exponent.

Note that

$$1 < m \leq \frac{\gamma + 1}{1 - \beta} < \frac{\gamma + 1}{1 - \alpha},$$

and

$$\gamma > -\beta > -\alpha.$$

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References

- [1] Agarwal R P, Benchohra M, Hamani S A. Survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math.*, 2010, 109, 973-1033.
- [2] Agarwal R P, Belmekki M, Benchohra M. A survey on semilinear differential equations and inclusions involving Riemann–Liouville fractional derivative. *Adv. Difference Equ.*, 2009, Article ID 981728, 1-47.
- [3] Bagley R L, Torvik P J. A theoretical basis for the application of fractional calculus to viscoelasticity. *J. Rheology*, 1983, 27, 201-210.
- [4] Bagley R L, Torvik P J. A different approach to the analysis of viscoelastically damped structures. *AIAA Journal*, 1983, 21, 741-748.
- [5] Bagley R L, Torvik P J. On the appearance of the fractional derivative in the behavior of real material. *J. Appl. Mechanics*, 1983, 51, 294-298.
- [6] Cascaval R C, Eckstein E C, Frota C L, Godstein J A. Fractional telegraph equations. *J. Math. Anal. Appl.*, 2002, 276, 145-159.
- [7] Furati K F, Tatar N E. An existence result for a nonlocal fractional differential problem. *J. Fract. Calc.*, 2004, 26, 43-51.
- [8] Furati K F, Tatar N E. Behavior of solutions for a weighted Cauchy-type fractional differential problem. *J. Fract. Calc.*, 2005, 28, 23-42.
- [9] Furati K F, Kassim M D, Tatar N E. Existence and uniqueness for a problem involving Hilfer fractional derivative. *Comput. Math. Appl.*, 2012, 64, 1616-1626.
- [10] Furati K F, Kassim M D, Tatar N E. Non-existence of global solutions for a differential equation involving Hilfer fractional derivative. *Electron. J. Diff. Equ.*, 2013, 2013, 1-10.
- [11] Hilfer R. *Fractional time evolution, Applications of fractional calculus in physics*. World Scientific, New-Jersey, London-Hong Kong, 2000, 87-130.
- [12] Kassim M D, Furati K F, Tatar N E. On a differential equation involving Hilfer-Hadamard fractional derivative. *Abstr. Appl. Anal.*, 2012, Article ID 391062, 1-17.
- [13] Kassim M D, Tatar N E. Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative. *Abstr. Appl. Anal.*, 2013, Article ID 605029, 1-12.
- [14] Kilbas A A, Srivastava H M, Trujillo J J. *Theory and Applications of Fractional Differential Equations*, Elsevier Science, 2006, 204.
- [15] Kirane M, Medved M, Tatar N E. On the nonexistence of blowing-up solutions to a fractional functional differential equations. *Georgian J. Math.*, 2012, 19, 127-144.
- [16] Kirane M, Tatar N E. Nonexistence of solutions to a hyperbolic equation with a time fractional damping. *Z. Anal. Anwendungen*, 2006, 25, 131-142.
- [17] Kirane M, Tatar N E. Absence of local and global solutions to an elliptic system with time-fractional dynamical boundary conditions. *Siberian J. Math.*, 2007, 48, 477-488.
- [18] Kirane M, Laskri Y, Tatar N E. Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives. *J. Math. Anal. Appl.*, 2005, 312, 488-501.
- [19] Kiryakova V. *Generalized Fractional Calculus and Applications*. John Wiley & Sons Inc., 1994, New York.
- [20] Koeller R C. Application of fractional calculus to the theory of viscoelasticity. *J. Appl. Mechanics*, 1984, 51, 299-307.
- [21] Laskri Y, Tatar N E. The critical exponent for an ordinary fractional differential problem. *Comput. Math. Appl.*, 2010, 59, 1266-1270.

- [22] Mainardi F, Gorenflo R. Time-fractional derivatives in relaxation processes: a tutorial survey. *Fract. Calc. Appl. Anal.*, 2007, 10, 269–308.
- [23] Mainardi F. *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, 2010, London.
- [24] Messaoudi S A, Said-Houari B, Tatar N E. Global existence and asymptotic behavior for a fractional differential equation. *Appl. Math. Comput.*, 2007, 188, 1955–1962.
- [25] Miller K S, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley, 1993, New York.
- [26] Oldham K B, Spanier J. *The Fractional Calculus*. Academic Press, 1974, New York, London.
- [27] Podlubny I. *Fractional Differential Equations, Mathematics in Sciences and Engineering*. Academic Press, 1999, San-Diego.
- [28] Podlubny I. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calcul. Anal. Appl.*, 2002, 5, 367–386.
- [29] Podlubny I, Petráš I, Vinagre B M, O’Leary P, Dorčák L. Analogue realizations of fractional-order controllers. *Nonlinear Dynam.*, 2002, 29, 281–296.
- [30] Tatar N E. Nonexistence results for a fractional problem arising in thermal diffusion in fractal media. *Chaos Solitons Fractals*, 2008, 36, 1205–1214.
- [31] Tatar N E. Existence results for an evolution problem with fractional nonlocal conditions. *Comput. Math. Appl.*, 2010, 60, 2971–2982.